

Improved Recovery Guarantees for Phase Retrieval from Coded Diffraction Patterns

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ABSTRACT. In this work we analyze the problem of phase retrieval from Fourier measurements with random diffraction patterns. To this end, we consider the recently introduced PhaseLift algorithm, which expresses the problem in the language of convex optimization. We provide recovery guarantees which require $\mathcal{O}(\log^2 d)$ different diffraction patterns, thus improving on recent results by Candès et al. [1], which require $\mathcal{O}(\log^4 d)$ different patterns.

1. INTRODUCTION

1.1. The problem of phase retrieval. In this work we are interested in the problem of *phase retrieval* which is of considerable importance in many different areas of science, where capturing phase information is hard or even infeasible. Problems of this kind occur, for example, in X-ray crystallography, diffraction imaging, and astronomy.

More formally, *phase retrieval* is the problem of recovering an unknown complex vector $x \in \mathbb{C}^d$ from an *intensity* measurement $y_0 = \|x\|_{\ell_2}^2$ and *amplitude* measurements

$$(1) \quad y_i = |\langle a_i, x \rangle|^2 \quad i = 1, \dots, m,$$

for a given set of measurement vectors $a_1, \dots, a_m \in \mathbb{C}^d$. The observations y are insensitive to a global phase change $x \mapsto e^{i\phi}x$ – hence in the following, notions like “recovery” or “injectivity” are always implied to mean “up to a global phase”. Clearly, the most fundamental question is: Which families of measurement vectors $\{a_i\}$ allow for a recovery of x in principle? I.e., for which measurements is the map $x \mapsto y$ defined by (1) injective?

Approaches based on algebraic geometry (for example [2, 3]) have established that $4d + o(1)$ *generic* measurements are both necessary and sufficient to determine x . Here, “generic” means that the measurement ensembles for which the property fails to hold lie on a low-dimensional subvariety of the algebraic variety of all tight measurement frames.

This notion of generic success, however, is mainly of theoretical interest. Namely, injectivity alone neither gives an indication on how to recover the unique solution, nor is there any chance to directly generalize the results to the case of noisy measurements. It should be noted, however, that recently the notion of injectivity has been refined to capture aspects of stability with respect to noise [4].

Paralleling these advances, there have been various attempts to find tractable recovery algorithms that yield recovery guarantees. Many of these approaches are based on a linear reformulation in matrix space, which is well-known in convex programming. The crucial underlying observation is that the quadratic constraints (1) on x are linear in the outer product $X = xx^*$:

$$y_i = |\langle a_i, x \rangle|^2 = \text{tr}((a_i a_i^*)X).$$

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Balan et al. [5] observed that for the right choice of d^2 measurement vectors a_i , this linear system in the entries of X admits for a unique solution, so the problem can be explicitly solved using linear algebra techniques. This approach, however, does not make use of the low-rank structure of X , which is why the required number of measurements is so much larger than what is required for injectivity.

The *PhaseLift* algorithm proposed by Candès et al. [6, 7, 8] uses in addition the property that X is of rank one, so even when the number of measurements is smaller than d^2 and there is an entire affine space of matrices satisfying (1.1), X is the solution of smallest rank. While finding the smallest rank solution of a linear system is, in general, NP hard, there are a number of algorithms known to recover the smallest rank solution provided the system satisfies some regularity conditions. The first such results were based on convex relaxation (see, for example, [9, 10, 11]). *PhaseLift* is also based on this strategy. For measurement vectors drawn independently at random from a Gaussian distribution, the number of measurements required to guarantee recovery with high probability was shown to be of optimal order, scaling linearly in the dimension [7, 8]. Ref. [12] even identifies a deterministic, explicitly engineered set of $4d - 4$ measurement vectors and proves that *PhaseLift* will successfully recover generic signals from the associated measurements.

Since these first recovery guarantees for the phase retrieval problem, recovery guarantees have been proved for a number of more efficient algorithms closer to the heuristic approaches typically used in practice. For example, in [13], an approach based on polarization is analyzed and in [14], the authors study an alternating minimization algorithm. In both works, recovery guarantees are again proved for Gaussian measurements.

To relate all these results to practice, the structure of applications needs to be incorporated into the setup, which corresponds to reducing randomness and considering structured measurements. For *PhaseLift*, the first partial derandomization has been provided by the authors of this paper, considering measurements sampled from spherical designs, that is, polynomial-size sets which generalize the notion of a tight frame to higher-order tensors [15]. Arguably, these derandomized measurement setups are still mainly of theoretical interest.

A structured measurement setup closer to applications is that of coded diffraction patterns. These correspond to the composition of diagonal matrices and the Fourier transform and model the modified application setup where diffraction masks are placed between the object and the screen as originally proposed in [16]. The first recovery guarantees from masked Fourier measurements were provided for polarization based recovery [17], where the design of the masks is very specific and intimately connected to the recovery algorithm. The required number of masks is $\mathcal{O}(\log(d))$, which corresponds to $\mathcal{O}(d \log(d))$ measurements.

For the *PhaseLift* algorithm, recovery guarantees from masked Fourier measurements were first provided in [1]. The results require $\mathcal{O}(d \log^4 d)$ measurements and hold with high probability when the masks are chosen at random, which is in line with the observation from [16] that random diffraction patterns are particularly suitable.

In this paper, we consider the same measurement setup as [1], but improve the bound on the required number of measurements to $\mathcal{O}(d \log^2 d)$.

2. PROBLEM SETUP AND MAIN RESULTS

2.1. Coded diffraction patterns. As in [1], we will work with the following setup:

In every step, we collect the magnitudes of the discrete Fourier transform (DFT) of a random modulation of the unknown signal x . Each such modulation pattern is modeled

by a random diagonal matrix. More formally, for $\omega := \exp\left(\frac{2\pi i}{d}\right)$ a d -th root of unity and $\{e_1, \dots, e_d\}$ the standard basis of \mathbb{C}^d , denote by

$$(2) \quad f_k = \sum_{j=1}^d \omega^{jk} e_j$$

the k -th discrete Fourier vector, normalized so that each entry has unit modulus. Furthermore, consider the diagonal matrix

$$(3) \quad D_l = \sum_{i=1}^d \epsilon_{l,i} e_i e_i^*$$

where the $\epsilon_{l,i}$'s are independent copies of a random variable ϵ which obeys

$$\mathbb{E}[\epsilon] = \mathbb{E}[\epsilon^3] = 0,$$

$$(4) \quad |\epsilon| \leq b \quad \text{almost surely for some } b > 0,$$

$$(5) \quad \mathbb{E}[\epsilon^4] = 2 \mathbb{E}[\epsilon^2]^2 \quad \text{and we define } \nu := \mathbb{E}[\epsilon^2].$$

Then the measurements are given by

$$(6) \quad y_{k,l} = |\langle f_k, D_l x \rangle|^2 \quad 1 \leq k \leq d, \quad 1 \leq l \leq L.$$

It turns out (Lemma 6 below) that condition (5) on ϵ ensures that the measurement ensemble forms a spherical 2-design, which draws a connection to [5] and [15].

As an example, the criteria above include the model

$$\epsilon \sim \begin{cases} \sqrt{2} & \text{with prob. } 1/4, \\ 0 & \text{with prob. } 1/2, \\ -\sqrt{2} & \text{with prob. } 1/4. \end{cases}$$

which has been discussed in [1]². In this case, each modulation is given by a Rademacher vector with random erasures.

2.2. Convex Relaxation. Following [5], we rewrite the measurement constraints as the inner product of two rank 1 matrices, one representing the signal, the other one the measurement coefficients. In the coded diffraction setup, we obtain, as in [1], the inner product of (6) can be translated into matrix form by applying the following “lifts”:

$$X := x x^* \quad \text{and} \quad F_{k,l} := D_l f_k f_k^* D_l.$$

Occasionally, we will make use of the representation with respect to the standard basis, which reads

$$(7) \quad F_{k,l} = \sum_{i,j=1}^d \epsilon_{l,i} \epsilon_{l,j} \omega^{k(i-j)} e_i e_j^*.$$

With these definitions, the dL individual linear measurements assume the following form

$$y_{k,l} = \text{Tr}(F_{k,l} X) \quad k = 1, \dots, d, \quad 1 \leq l \leq L.$$

and the phase retrieval problem thus becomes the problem of finding rank 1 solutions $X = x x^*$ compatible with these affine constraints. Rank-minimization over affine spaces

² Ref. [1] also included a strongly related model where ϵ is a complex random variable. We have opted to keep ϵ real, which implies that the D_l are Hermitian. This, in turn, has allowed us to slightly simplify notation throughout.

is NP-hard in general. However, it is now well-appreciated [9, 10, 11, 7] that nuclear-norm based convex relaxations solve this problems efficiently in many relevant instances. Applied to phase retrieval, the relaxation becomes

$$(8) \quad \begin{aligned} \operatorname{argmin}_{X'} \quad & \|X'\|_1 \\ \text{subject to} \quad & \operatorname{tr}(F_{k,l}X') = y_{k,l} \quad k = 1, \dots, n, 1 \leq l \leq L, \\ & X' = (X')^* \\ & \operatorname{tr}(X') = 1, \\ & X' \geq 0, \end{aligned}$$

which has been dubbed *Phaselift* by its inventors [6, 7, 8]. For this convex relaxation, recovery guarantees are known for measurement vectors drawn i.i.d. at random from a Gaussian distribution [7, 8], t -designs [15], or in the masked Fourier setting [1].

2.3. Our contribution. In this paper, we adopt the setup from [1]. Our main message is recovery of x can be guaranteed already for

$$L \geq C \log^2 d$$

random diffraction patterns, which improves the bound given in [1] by a factor of $\mathcal{O}(\log^2 d)$. This is significant, as it indicates that the provably achievable rates are approaching the ultimate limit. Indeed, for the Rademacher masks with random erasures introduced above, a lower bound for the number of measurements required to allow for recovery with any algorithm is given by $\mathcal{O}(\log d)$. (To see this, assume that the signal equals the first standard basis vector e_1 . Then for $o(\log(d))$ masks, a coupons collector argument similar to the ones provided in [10, 11] yields that there is another standard basis vector with an identical associated measurement vector.) Thus there cannot be a recovery algorithm requiring fewer than $\mathcal{O}(\log(d))$ masks and there is only a single log-factor separating our results from an asymptotically tight solution.

More precisely, our version of [1, Theorem 1.1] reads:

Theorem 1 (Main Theorem). *Let $x \in \mathbb{C}^d$ be an unknown signal with $\|x\|_{\ell_2} = 1$ and let $d \geq 3$ be an odd number. Suppose that L complete Fourier measurements using independent random diffraction patterns (as defined in Section 2.1) are performed.*

Then with probability at least $(1 - e^{-\omega})$ Phaselift (the convex optimization problem (8)) recovers x up to a global phase, provided that

$$L \geq C\omega \log^2 d.$$

Here, $\omega \geq 1$ is an arbitrary parameter and C a dimension-independent constant that can be explicitly bounded.

The number C is of the form $C = \tilde{C} \frac{b^8}{\nu^4} \log \frac{b^2}{\nu}$, with \tilde{C} an absolute constant for which an explicit estimate can be extracted from our proof.

For the benefit of the technically minded reader, we briefly sketch the relation between the proof techniques used here, as compared to References [1] and [15].

- The general structure of this document closely mimics [15] (which bears remarkable similarity to [1], even though the papers were written completely independently and with different aims in mind).
- From [1] we borrow the use of Hoeffding's inequality to bound the probability of "the inner product between the measurement vectors and the signal becoming too large". This is Lemma 12 below. Our previous work also bounded the probability

of such events [15, Lemma 13] – however in a weaker way (relying only on certain t th moments as opposed to a Hoeffding bound).

- Both [15, 1] as well as the present paper estimate the condition number of the measurement operator restricted to the tangent space at xx^* (“robust injectivity”). Our Proposition 7 improves over [1, Section 3.3] by using an operator Bernstein inequality instead of a weaker operator Hoeffding bound.
- Finally, we use a slightly refined version of the golfing scheme to construct an approximate dual certificate (following [11, Section III.B]).

2.4. More general bases and outlook. The result allows for a fairly general distribution of the masks D_l , but refers specifically to the Fourier basis. An obvious question is how sensitively the statements depend on the properties of this basis.

We begin by pointing out that Theorem 1 immediately implies a corollary for higher-dimensional Fourier transforms. In diffraction imaging applications, for example, one would naturally employ a 2-D Fourier basis

$$(9) \quad f_{k,l} = \sum_{i=1}^{d_x} \sum_{j=1}^{d_y} \omega_{d_x}^{ik} \omega_{d_y}^{jl} e_{i,j},$$

with d_x and d_y the horizontal and vertical resolution respectively, $\omega_d := \exp\left(\frac{2\pi i}{d}\right)$, and $e_{i,j}$ the position space basis vector representing a signal located at coordinates (i, j) . Superficially, (9) looks quite different from the one-dimensional case (2). However, a basic application of the Chinese Remainder Theorem shows that if d_x and d_y are co-prime, then the 2-D transform reduces to the 1-D one for dimension $d_x d_y$ (in the sense that the respective bases agree up to relabeling) [18]. An analogous result holds for higher-dimensional transforms [18], proving the following corollary.

Corollary 2. *Assume $d = \prod_{i=1}^k d_i$ is the product of mutually co-prime odd numbers greater than 3. Then Theorem 1 remains valid for the k -dimensional Fourier transform over d_1, \dots, d_k .*

More generally speaking, our argument employs the particular properties of Fourier bases in two places: Lemma 6 and Lemma 8.

The former lemma shows that the measurements are drawn from an *isotropic ensemble* (or *tight frame*) in the relevant space of Hermitian matrices. A similar condition is frequently used in works on phase retrieval, low-rank matrix completion, and compressed sensing (e.g. [15, 1, 19, 20, 11]). Properties of the Fourier basis are used in the proof of Lemma 6 only for concreteness. Using relatively straight-forward representation theory, one can give a far more abstract version of the result which is valid for any basis satisfying two explicit polynomial relations (cf. the remark below the lemma). The combinatorial structure of Fourier transforms is immaterial at this point.

This contrasts with Lemma 8 which currently prevents us from generalizing the main result to a broader class of bases. Its proof uses explicit coordinate expressions of the Fourier basis to facilitate a series of simplifications. Identifying the abstract gist of the manipulations is the main open problem which we hope to address in future work.

It would also be interesting to use the techniques of the present paper to re-visit the problem of quantum state tomography [21, 22, 23, 24] (which was the initial motivation for one of the authors to become interested in low-rank recovery methods). Indeed, the original work on quantum state tomography and low-rank recovery [21] was based on a model where the expectation value of a Pauli matrix is the elementary unity of information

extractable from a quantum experiment. While this correctly describes some experiments, it is arguably more common that the statistics of the eigenbasis of an observable are the objects that can be physically directly accessed. For this practically more relevant case, no recovery guarantees seem to be currently known and the methods used here could be used to ammend that situation.

3. TECHNICAL BACKGROUND AND NOTATION

3.1. Vectors, Matrices and matrix valued Operators. The signals x are assumed to live in \mathbb{C}^d equipped with the usual inner product $\langle \cdot, \cdot \rangle$. We denote the induced norm by

$$\|z\|_{\ell_2} = \sqrt{\langle z, z \rangle} \quad \forall z \in \mathbb{C}^d.$$

Vectors in \mathbb{C}^d will be denoted by lower case Latin characters. For $z \in \mathbb{C}^d$ we define the absolute value $|z| \in \mathbb{R}_+^d$ component-wise $|z|_i = |z_i|$.

On the level of matrices we will exclusively encounter $d \times d$ hermitian matrices and denote them by capital Latin characters. Endowed with the Hilbert-Schmidt (or Frobenius) scalar product

$$(10) \quad (Z, Y) = \text{tr}(ZY)$$

the space H^d of all $d \times d$ hermitian matrices becomes a Hilbert space itself. In addition to that, we will require three different operator norms

$$(11) \quad \begin{aligned} \|Z\|_1 &= \text{tr}(|Z|) \quad (\text{trace or nuclear norm}), \\ \|Z\|_2 &= \sqrt{\text{tr}(Z^2)} \quad (\text{Frobenius norm}), \\ \|Z\|_\infty &= \sup_{y \in \mathbb{C}^d} \frac{|\langle y, Zy \rangle|}{\|y\|_{\ell_2}^2} \quad (\text{operator norm}). \end{aligned}$$

For arbitrary matrices Z of rank at most r , the norms above are related via the inequalities

$$\|Z\|_2 \leq \|Z\|_1 \leq \sqrt{r}\|Z\|_2 \quad \text{and} \quad \|Z\|_\infty \leq \|Z\|_2 \leq \sqrt{r}\|Z\|_\infty.$$

Recall that a hermitian matrix Z is positive semidefinite if one has $\langle y, Zy \rangle \geq 0$ for all $y \in \mathbb{C}^d$. We write $Y \geq Z$ iff $Y - Z$ is positive semidefinite.

In this work, hermitian rank-1 projectors are of particular importance. They are of the form $Z = zz^*$ with $z \in \mathbb{C}^d$. The vector z can then be recovered from Z up to a global phase factor via the singular value decomposition. The rank-1 projectors most relevant to this work $X = xx^*$ and $F_{k,l} = D_l f_k (D_l f_k)^*$.

Finally, we will also encounter *matrix-valued operators* acting on the matrix space H^d . Here, we will restrict ourselves to operators that are hermitian with respect to the Hilbert-Schmidt inner product. We label such objects with calligraphic letters. The operator norm becomes

$$(12) \quad \|\mathcal{M}\|_{\text{op}} = \sup_{Z \in H^d} \frac{|\text{tr}(Z\mathcal{M}Z)|}{\|Z\|_2^2}.$$

It turns out that only two classes of such operators will appear in our work, namely the identity map

$$\begin{aligned} \mathcal{I} : H^d &\rightarrow H^d \\ Z &\mapsto Z \quad \forall Z \in H^d \end{aligned}$$

and (scalar multiples of) projectors onto some matrix $Y \in H^d$ as given by

$$\begin{aligned}\Pi_Y : H^d &\rightarrow H^d \\ Z &\mapsto Y(Y, Z) = Y \operatorname{tr}(YZ) \quad \forall Z \in H^d.\end{aligned}$$

An important example of the latter class is

$$\Pi_{\mathbb{1}} : Z \mapsto \mathbb{1} \operatorname{tr}(\mathbb{1}Z) = \operatorname{tr}(Z) \mathbb{1} \quad \forall Z \in H^d.$$

Note that the normalization is such that $\frac{1}{d}\Pi_{\mathbb{1}}$ is idempotent, i.e. a properly normalized projection. Indeed, for $Z \in H^d$ arbitrary it holds that

$$(13) \quad (d^{-1}\Pi_{\mathbb{1}})^2 Z = d^{-2} \mathbb{1} \operatorname{tr}(\mathbb{1}\Pi_{\mathbb{1}} Z) = d^{-2} \operatorname{tr}(\mathbb{1}) \operatorname{tr}(Z) \mathbb{1} = d^{-1}\Pi_{\mathbb{1}} Z.$$

The notion of positive-semidefiniteness directly translates to matrix valued operators. It is easy to check that all the operators introduced so far are positive semidefinite. From (13) we obtain the ordering

$$(14) \quad 0 \leq \Pi_{\mathbb{1}} \leq d\mathcal{I}.$$

3.2. Tools from Probability Theory. In this section, we recall some concentration inequalities which will prove useful for our argument. Our first tool is a variant of Hoeffding's inequality [25].

Theorem 3. *Let $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ be an arbitrary vector and let $\epsilon_i, i = 1, \dots, d$, be independent copies of a centered random variable ϵ which is almost surely bounded in modulus by $b > 0$. Then*

$$(15) \quad \Pr \left[\left| \sum_{i=1}^d \epsilon_i z_i \right| \geq t \|z\|_2 \right] \leq 2 \exp(-t^2/(2b^2)).$$

Secondly, we will require two matrix versions of Bernstein's inequality. Such matrix valued large deviation bounds have been established first in the field of quantum information by Ahlswede and Winter [26] and introduced to sparse and low-rank recovery in [21, 11]. We make use of refined versions from [27, 28]. Note that as H^d is a finite dimensional vector space, the results also apply to matrix valued operators as introduced in section 3.1.

Theorem 4 (Uniform Operator Bernstein inequality, [27, 11]). *Consider a finite sequence $\{M_k\}$ of independent random self-adjoint matrices. Assume that each M_k satisfies $\mathbb{E}[M_k] = 0$ and $\|M_k\|_{\infty} \leq \bar{R}$ (for some finite constant \bar{R}) almost surely. Then with the variance parameter $\sigma^2 := \|\sum_k \mathbb{E}[M_k^2]\|_{\infty}$, the following chain of inequalities holds for all $t \geq 0$.*

$$\Pr \left[\left\| \sum_k M_k \right\|_{\infty} \geq t \right] \leq d \exp \left(-\frac{t^2/2}{\sigma^2 + \bar{R}t/3} \right) \leq \begin{cases} d \exp(-3t^2/8\sigma^2) & t \leq \sigma^2/\bar{R} \\ d \exp(-3t/8\bar{R}) & t \geq \sigma^2/\bar{R}. \end{cases}$$

Theorem 5 (Smallest Eigenvalue Bernstein Inequality, [28]). *Let $S = \sum_k M_k$ be a sum of i.i.d. random matrices M_k which obey $\mathbb{E}[M_k] = 0$ and $\lambda_{\min}(M_k) \geq -\underline{R}$ almost surely for some fixed \underline{R} . With the variance parameter $\sigma^2(S) = \|\sum_k \mathbb{E}[M_k^2]\|_{\infty}$ the following chain of inequalities holds for all $t \geq 0$.*

$$\Pr[\lambda_{\min}(S) \leq -t] \leq d \exp \left(-\frac{t^2/2}{\sigma^2 + \underline{R}t/3} \right) \leq \begin{cases} d \exp(-3t^2/8\sigma^2) & t \leq \sigma^2/\underline{R} \\ d \exp(-3t/8\underline{R}) & t \geq \sigma^2/\underline{R}. \end{cases}$$

4. PROOF INGREDIENTS

4.1. **Near-isotropy.** In this section we study the *measurement operator*³

$$(16) \quad \mathcal{R} : H^d \rightarrow H^d, \quad \mathcal{R} := \sum_{l=1}^L \mathcal{M}_l \quad \text{with}$$

$$(17) \quad \mathcal{M}_l Z := \frac{1}{\nu^2 d L} \sum_{k=1}^d \Pi_{F_{k,l}} Z = \frac{1}{\nu^2 d L} \sum_{k=1}^d F_{k,l} \operatorname{tr}(F_{k,l} Z),$$

which just corresponds to $\mathcal{R} = \frac{1}{\nu^2 d L} \mathcal{A}^* \mathcal{A}$, where ν was defined in (5).

The following result shows that this operator is *near-isotropic* in the sense of [15, 6].

Lemma 6 (\mathcal{R} is *near-isotropic*). *The operator \mathcal{R} defined in (16) is near-isotropic in the sense that*

$$(18) \quad \mathbb{E}[\mathcal{R}] = L \mathbb{E}[\mathcal{M}_l] = \mathcal{I} + \Pi_{\mathbb{1}} \quad \text{or} \quad \mathbb{E}[\mathcal{R}(Z)] = Z + \operatorname{tr}(Z) \mathbb{1} \quad \forall Z \in H^d.$$

A proof of Lemma 6 can be found in [1]. However, we still present a proof – which is of a slightly different spirit – in the appendix for the sake of being self-contained.

Two remarks are in order with regard to the previous lemma.

First, it is worthwhile to point out that *near-isotropy* of \mathcal{R} is equivalent to stating that the set of all possible realizations of $D_l f_k$ form a 2-design. This has been made explicit recently in [29, Lemma 1]. The notion of higher-order spherical designs is the basic mathematical object of our previous work [15] on phase retrieval.

Second, our proof of Lemma 6 uses the explicit representation of the measurement vectors with respect to the standard basis. As alluded to in Section 2.4, a more abstract proof can be given. We sketch the basic idea here and refer the reader to an upcoming work for details [30]. Consider the case where ϵ is a symmetric random variable (i.e., where ϵ has the same distribution as $-\epsilon$). In that case, the distribution of the D_l is plainly invariant under permutations of the main diagonal elements and under element-wise sign changes. These are the symmetries of the d -cube. They constitute the group $\mathbb{Z}_2^d \rtimes S_d$, sometimes referred to as the *hyperoctahedral group*. Using a standard technique [31, 32], conditions for near-isotropy can be phrased in terms of the representation theory of the hyperoctahedral group acting on $\operatorname{Sym}^2(\mathbb{C}^d)$. This action decomposes into three explicitly identifiable irreducible components, from which one can deduce that near-isotropy holds for any basis that fulfills two 4th order polynomial equations [30].

Let now $x \in \mathbb{C}^d$ be the signal we aim to recover. Since the intensity of x (i.e., its ℓ_2 -norm) is known by assumption, we can w.l.o.g. assume that $\|x\|_{\ell_2} = 1$. As in [7, 15, 1] we consider the space

$$(19) \quad T := \{xz^* + zx^* : z \in \mathbb{C}^d\} \subset H^d$$

which is the tangent space of the manifold of all rank-1 hermitian matrices at the point $X = xx^*$. The orthogonal projection onto this space can be given explicitly:

$$(20) \quad \mathcal{P}_T : H^d \rightarrow H^d$$

$$(21) \quad \begin{aligned} Z &\mapsto XZ + ZX - XZX \\ &= XZ + ZX - \operatorname{tr}(XZ)X. \end{aligned}$$

³ We are going to use the notations $\mathcal{M}(Z)$ and $\mathcal{M}Z$ equivalently.

The Frobenius inner product allows us to define an ortho-complement T^\perp of T in H^d . We denote the projection onto T^\perp by \mathcal{P}_T^\perp and decompose any matrix $Z \in H^d$ as

$$Z = \mathcal{P}_T Z + \mathcal{P}_T^\perp Z =: Z_T + Z_T^\perp.$$

We point out that, in particular,

$$(22) \quad \mathcal{P}_T \Pi_1 \mathcal{P}_T = \Pi_X \quad \text{and} \quad \|\mathcal{P}_T Z\|_\infty \leq 2\|Z\|_\infty$$

holds for any $Z \in H^d$. The first fact follows by direct calculation, while the second one comes from

$$\|Z_T\|_\infty = \|Z - Z_T^\perp\|_\infty \leq \|Z\|_\infty + \|Z_T^\perp\|_\infty \leq 2\|Z\|_\infty$$

where the last estimate used the pinching inequality [33] (Problem II.5.4).

4.2. Well-posedness/Injectivity. In this section, we follow [7, 11, 1] in order to establish a certain injectivity property of the measurement operator \mathcal{A} .

Our Proposition 7 is the analogue of Lemma 3.7 in [1]. The latter contained an factor of $\mathcal{O}(\log^2 d)$ in the exponent of the failure probability, which does not appear here. The reason is that we employ a single-sided Bernstein inequality, instead of a symmetric Hoeffding inequality.

Proposition 7 (Robust injectivity, lower bound). *With probability of failure smaller than $d^2 \exp\left(-\frac{\nu^4 L}{C_1 b^8}\right)$ the inequality*

$$(23) \quad \frac{1}{\nu^2 d L} \|\mathcal{A}(Z)\|_{\ell_2}^2 > \frac{1}{4} \|Z\|_2^2$$

is valid for all matrices $Z \in T$ simultaneously. Here b and ν are as in (4, 5) and C_1 is an absolute constant.

We require bounds on certain variances for the proof of this statement. The technical Lemma 8 serves this purpose.

Lemma 8. *Let $Z \in T$ be an arbitrary matrix and let \mathcal{M}_l be as in (17). Then it holds that*

$$(24) \quad \|\mathbb{E} [\mathcal{M}_l(Z)^2]\|_\infty \leq \frac{30b^8}{\nu^4 L^2} \|Z\|_2^2,$$

and

$$(25) \quad \|\mathbb{E} [(\mathcal{P}_T \mathcal{M}_l(Z))^2]\|_\infty \leq \frac{90b^8}{\nu^4 L^2} \|Z\|_2^2.$$

In the following proof we will use that for $a, b \in \mathbb{Z}_d = \{0, \dots, d-1\}$ one has

$$(26) \quad \frac{1}{d} \sum_{k=1}^d \omega^{k(a \oplus b)} = \delta_{a,b} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{else.} \end{cases}$$

The symbols \oplus and \ominus denote addition and subtraction modulo d .

Proof of Lemma 8. Let $y, z, v \in \mathbb{C}^d$ be vectors of unit length. Compute:

$$\begin{aligned}
(27) &= \nu^4 L^2 \mathbb{E} [\mathcal{M}(yy^*) \mathcal{M}(zz^*)] v \\
&= \frac{1}{d^2} \sum_{k,j=1}^d \mathbb{E} \left[\left(\sum_{i_3, i_4=1}^d \epsilon_{i_3} \epsilon_{i_4} \omega^{k(i_3-i_4)} \bar{y}_{i_3} y_{i_4} \right) \left(\sum_{i_5, i_6=1}^d \epsilon_{i_5} \epsilon_{i_6} \omega^{j(i_5-i_6)} \bar{z}_{i_5} z_{i_6} \right) \right. \\
&\quad \times \left. \sum_{i_1, i_2, i_7, i_8=1}^d \epsilon_{i_1} \epsilon_{i_2} \omega^{k(i_2-i_1)} \epsilon_{i_7} \epsilon_{i_8} \omega^{j(i_8-i_7)} e_{i_2} \delta_{i_1, i_8} v_{i_7} \right] \\
&= \sum_{i_1, \dots, i_7} \mathbb{E} [\epsilon_{i_1}^2 \epsilon_{i_2} \dots \epsilon_{i_7}] \left(\frac{1}{d} \sum_k \omega^{k(i_2+i_3-i_1-i_4)} \right) \left(\frac{1}{d} \sum_j \omega^{j(i_5+i_1-i_6-i_7)} \right) \\
&\quad \times \bar{y}_{i_3} y_{i_4} \bar{z}_{i_5} z_{i_6} v_{i_7} e_{i_2} \\
(28) &= \sum_{i_1, \dots, i_7} \mathbb{E} [\epsilon_{i_1}^2 \epsilon_{i_2} \dots \epsilon_{i_7}] \delta_{i_1, (i_2 \oplus i_3 \ominus i_4)} \delta_{i_1, (i_6 \oplus i_7 \ominus i_5)} \bar{y}_{i_3} y_{i_4} \bar{z}_{i_5} z_{i_6} v_{i_7} e_{i_2} \\
(29) &= \sum_{i_2, \dots, i_7} \mathbb{E} [\epsilon_{i_2 \oplus i_3 \ominus i_4}^2 \epsilon_{i_2} \dots \epsilon_{i_7}] \delta_{i_2, (i_4 \oplus i_6 \oplus i_7 \ominus i_3 \ominus i_5)} \bar{y}_{i_3} y_{i_4} \bar{z}_{i_5} z_{i_6} v_{i_7} e_{i_2},
\end{aligned}$$

where in (27) we have inserted the definition of \mathcal{M}_l , in (28) have made use of (26), and in (29) we have eliminated i_1 . We now make the crucial observation that the expectation

$$(30) \quad \mathbb{E} [\epsilon_{i_2 \oplus i_3 \ominus i_4}^2 \epsilon_{i_2} \dots \epsilon_{i_7}]$$

vanishes unless every number in i_2, \dots, i_7 appears at least twice. More formally, the expectation is zero unless the set $\{2, \dots, 7\}$ can be partitioned into a disjoint union of pairs $\{2, \dots, 7\} = \bigcup_{\{k, l\} \in E} \{k, l\}$ such that $i_k = i_l$ for every $\{k, l\} \in E$ (in graph theory, E would be a set of edges constituting a *matching*). Indeed, assume to the contrary that there is some j such that i_j is unmatched (i.e., $i_j \neq i_k$ for all $k \neq j$). We distinguish two cases: If $i_j \neq i_2 \oplus i_3 \ominus i_4$, then ϵ_j appears only once in the product in (30) and the expectation vanishes because $\mathbb{E}[\epsilon_j] = 0$ by assumption. If $i_j = i_2 \oplus i_3 \ominus i_4$, then the same conclusion holds because we have also assumed that $\mathbb{E}[\epsilon_j^3] = 0$ (this is the only point in the argument where we need third moments of ϵ to vanish).

With this insight, we can proceed to put a tight bound on the ℓ_2 -norm of the initial expression.

$$\begin{aligned}
&\|\nu^4 L^2 \mathbb{E} [\mathcal{M}(yy^*) \mathcal{M}(zz^*)] v\|_{\ell_2} \\
&= \left\| \sum_{i_2, \dots, i_7=1}^d \mathbb{E} [\epsilon_{i_2 \oplus i_3 \ominus i_4}^2 \epsilon_{i_2} \dots \epsilon_{i_7}] \delta_{i_2, (i_4 \oplus i_6 \oplus i_7 \ominus i_3 \ominus i_5)} \bar{y}_{i_3} y_{i_4} \bar{z}_{i_5} z_{i_6} v_{i_7} e_{i_2} \right\|_{\ell_2} \\
&\leq \left\| \sum_{i_2, \dots, i_7=1}^d \mathbb{E} [\epsilon_{i_2 \oplus i_3 \ominus i_4}^2 \epsilon_{i_2} \dots \epsilon_{i_7}] \bar{y}_{i_3} y_{i_4} \bar{z}_{i_5} z_{i_6} v_{i_7} e_{i_2} \right\|_{\ell_2} \\
&\leq \sum_{\text{matchings } E} \left\| \sum_{\substack{i_2, \dots, i_7 \\ i_k = i_l \text{ for } \{k, l\} \in E}} \mathbb{E} [\epsilon_{i_2 \oplus i_3 \ominus i_4}^2 \epsilon_{i_2} \dots \epsilon_{i_7}] \bar{y}_{i_3} y_{i_4} \bar{z}_{i_5} z_{i_6} v_{i_7} e_{i_2} \right\|_{\ell_2} \\
(31) &\leq b^8 \sum_{\text{matchings } E} \left\| \sum_{\substack{i_2, \dots, i_7 \\ i_k = i_l \text{ for } \{k, l\} \in E}} \bar{y}_{i_3} y_{i_4} \bar{z}_{i_5} z_{i_6} v_{i_7} e_{i_2} \right\|_{\ell_2},
\end{aligned}$$

where the three inequalities follow, in that order, by realizing that making individual coefficients of e_{i_2} larger will increase the norm; restricting to non-zero expectation values as per the discussion above; and using the assumed bound $|\epsilon| \leq b$.

Now fix a matching E . Let $x^{(1)}$ be the vector in $\{v, \bar{y}, y, \bar{z}, z\}$ whose index in (31) is paired with i_2 . Label the remaining four vectors in that set by $x^{(2)}, \dots, x^{(5)}$, in such a way that $x^{(2)}$ and $x^{(3)}$ are paired and the same is true for $x^{(4)}$ and $x^{(5)}$. Then the summand corresponding to that matching becomes

$$\begin{aligned} & \left\| \sum_{a,b,c=1}^d \left| x_a^{(1)} x_b^{(2)} x_b^{(3)} x_c^{(4)} x_c^{(5)} \right| e_a \right\|_{\ell_2} \\ &= \left(\sum_{b=1}^d |x_b^{(2)} x_b^{(3)}| \right) \left(\sum_{c=1}^d |x_c^{(4)} x_c^{(5)}| \right) \left\| \sum_{a=1}^d |x_a^{(1)}| e_a \right\|_{\ell_2} \leq 1, \end{aligned}$$

by the Cauchy-Schwarz inequality and the fact that all the $x^{(i)}$ are of length one. As there are 15 possible matchings of 6 indices, we arrive at

$$\|\mathbb{E}[\mathcal{M}(yy^*)\mathcal{M}(zz^*)]v\|_{\ell_2} \leq \frac{15b^8}{\nu^4 L^2}.$$

Finally, let $Z \in T$. As Z has rank at most two, we can choose normalized vectors $y, z \in \mathbb{C}^d$ such that $Z = \lambda_1 yy^* + \lambda_2 zz^*$. Then

$$\|\mathbb{E}[\mathcal{M}(Z)^2]\|_{\infty} \leq \sum_{i,j=1}^2 \lambda_i \lambda_j \frac{15b^8}{\nu^4 L^2} = \|Z\|_1^2 \frac{15b^8}{\nu^4 L^2} \leq \|Z\|_2^2 \frac{30b^8}{\nu^4 L^2}.$$

For (25) we start by inserting (20) for \mathcal{P}_T , expanding the product and canceling terms using $X^2 = X$:

$$\begin{aligned} & \|\mathbb{E}[(\mathcal{P}_T \mathcal{M}_l(Z))^2]\|_{\infty} \\ &= \|\mathbb{E}[\mathcal{M}_l(Z)X\mathcal{M}_l(Z)] + X\mathbb{E}[\mathcal{M}_l(Z)^2]X - X\mathbb{E}[\mathcal{M}_l(Z)X\mathcal{M}_l(Z)]X\|_{\infty} \\ &\leq \|\mathbb{E}[\mathcal{M}_l(Z)\mathcal{M}_l(Z)]\|_{\infty} + 2\|\mathbb{E}[\mathcal{M}_l(Z)X\mathcal{M}_l(Z)]\|_{\infty}. \end{aligned}$$

For the latter term we can use $\mathcal{M}_L(Z)X\mathcal{M}_L(Z) \leq \mathcal{M}_l(Z)^2$. This follows from defining $Y^2 := \mathbb{1} - X \geq 0$ and observing that

$$\mathcal{M}_l(Z)^2 - \mathcal{M}_l(Z)X\mathcal{M}_l(Z) = (\mathcal{M}_l(Z)Y)(Y\mathcal{M}_l(Z)) \geq 0.$$

Using this observation together with monotonicity of expectation values and $\|\cdot\|_{\infty}$ reveals

$$\|\mathbb{E}[\mathcal{M}_l(Z)\mathcal{M}_l(Z)]\|_{\infty} + 2\|\mathbb{E}[\mathcal{M}_l(Z)X\mathcal{M}_l(Z)]\|_{\infty} \leq 3\|\mathbb{E}[\mathcal{M}_l(Z)\mathcal{M}_l(Z)]\|_{\infty}$$

and inequality (25) is thus implied by (24). \square

With Lemma 8 at hand, we can proceed to the lower bound on robust injectivity.

Proof of Proposition 7. We strongly follow the ideas presented in [15, Proposition 9] and aim to show the more general statement

$$(32) \quad \Pr[(\nu^2 dL)^{-1} \|\mathcal{A}(Z)\|_{\ell_2}^2 \leq (1 - \delta) \|Z\|_2^2 \quad \forall Z \in T] \leq d^2 \exp\left(-\frac{\nu^4 \delta^2 L}{\tilde{C}_1 b^8}\right)$$

for any $\delta \in (0, 1)$, where \tilde{C} is a numerical constant.

Pick $Z \in T$ arbitrary and use *near isotropicity* (18) of \mathcal{R} in order to write

$$\begin{aligned}
& (\nu^2 dL)^{-1} \|\mathcal{A}(Z)\|_{\ell_2}^2 \\
&= (\nu^2 dL)^{-1} \sum_{l=1}^L \sum_{k=1}^d (\text{tr}(F_{k,l}Z))^2 = \text{tr} \left(Z \frac{1}{\nu^2 dL} \sum_{l=1}^L \sum_{k=1}^d F_{k,l} \text{tr}(F_{k,l}Z) \right) \\
&= \text{tr}(Z\mathcal{R}Z) = \text{tr}(Z(\mathcal{R} - \mathbb{E}[\mathcal{R}])Z) + \text{tr}(Z(\mathcal{I} + \Pi_1)Z) \\
&= \text{tr}(Z\mathcal{P}_T(\mathcal{R} - \mathbb{E}[\mathcal{R}])\mathcal{P}_T Z) + \text{tr}(Z^2) + \text{tr}(Z)^2 \\
&\geq \text{tr}(Z\mathcal{P}_T(\mathcal{R} - \mathbb{E}[\mathcal{R}])\mathcal{P}_T Z) + \text{tr}(Z^2) \\
(33) \quad &\geq (1 + \lambda_{\min}(\mathcal{P}_T(\mathcal{R} - \mathbb{E}[\mathcal{R}])\mathcal{P}_T)) \|Z\|_2^2,
\end{aligned}$$

where we have used the fact that $\mathcal{M} \geq \lambda_{\min}(\mathcal{M})\mathcal{I}$ for any matrix valued operator \mathcal{M} as well as $\mathcal{P}_T Z = Z$. Therefore it suffices to bound the smallest eigenvalue of $\mathcal{P}_T(\mathcal{R} - \mathbb{E}[\mathcal{R}])\mathcal{P}_T$ from below. To this end we aim to use the Operator Bernstein inequality – Theorem 5 – and decompose

$$\mathcal{P}_T(\mathcal{R} - \mathbb{E}[\mathcal{R}])\mathcal{P}_T = \sum_{l=1}^L \left(\widetilde{\mathcal{M}}_l - \mathbb{E}[\widetilde{\mathcal{M}}_l] \right) \quad \text{with} \quad \widetilde{\mathcal{M}}_l = \mathcal{P}_T \mathcal{M}_l \mathcal{P}_T,$$

where \mathcal{M}_l was defined in (17). Note that these summands have mean zero by construction. Furthermore (22) implies

$$\begin{aligned}
-\frac{1}{\nu^2 L} \mathcal{I} - \frac{1}{\nu^2 L} \Pi_X &\leq -\frac{1}{\nu^2 L} \mathcal{P}_T \mathcal{I} \mathcal{P}_T - \frac{1}{\nu^2 L} \mathcal{P}_T \Pi_1 \mathcal{P}_T = -\frac{1}{\nu^2 L} \mathcal{P}_T \mathbb{E}[\mathcal{R}] \mathcal{P}_T \\
&= -\mathcal{P}_T \mathbb{E}[\mathcal{M}_l] \mathcal{P}_T \leq \widetilde{\mathcal{M}}_l - \mathbb{E}[\widetilde{\mathcal{M}}_l],
\end{aligned}$$

where the last inequality follows from $\widetilde{\mathcal{M}}_l \geq 0$. This yields an a priori bound

$$\lambda_{\min}(\widetilde{\mathcal{M}}_l - \mathbb{E}[\widetilde{\mathcal{M}}_l]) \geq -2/(\nu^2 L) =: -\underline{R}.$$

For the variance we use the standard identity

$$0 \leq \mathbb{E} \left[(\widetilde{\mathcal{M}}_l - \mathbb{E}[\widetilde{\mathcal{M}}_l])^2 \right] = \mathbb{E} \left[\widetilde{\mathcal{M}}_l^2 \right] - \mathbb{E} \left[\widetilde{\mathcal{M}}_l \right]^2 \leq \mathbb{E} \left[\widetilde{\mathcal{M}}_l^2 \right]$$

and focus on the last expression. For obtaining a bound on the total variance we are going to apply (12) to $\|\mathbb{E}[\widetilde{\mathcal{M}}_l^2]\|_{\text{op}}$. To this end, fix $Z \in T$ arbitrary – this restriction is valid, due to the particular structure of $\widetilde{\mathcal{M}}_l$ – and observe

$$\begin{aligned}
|\text{tr} \left(Z \mathbb{E} \left[\widetilde{\mathcal{M}}_l^2 \right] Z \right)| &= |\mathbb{E} [\text{tr} (\mathcal{M}_l(Z) \mathcal{P}_T \mathcal{M}_l(Z))]| = |\text{tr} (\mathbb{E} [(\mathcal{P}_T \mathcal{M}_l(Z))^2])| \\
&\leq 2 \|\mathbb{E} [(\mathcal{P}_T \mathcal{M}_l(Z))^2]\|_{\infty} \leq \frac{180b^8}{\nu^4 L^2} \|Z\|_2^2.
\end{aligned}$$

The first equality follows from inserting the definition (17) of \mathcal{M}_l and rewriting the expression of interest. For the second equality, we have used the fact that $\text{tr}(AB_T) = \text{tr}(A_T B_T)$ for any matrix pair $A, B \in H^d$ (\mathcal{P}_T is an orthogonal projection with respect to the Frobenius inner product) and the last estimate is due to (25) in Lemma 8. Since $Z \in T$ was arbitrary, we have obtained a bound on $\|\mathbb{E}[\widetilde{\mathcal{M}}_l^2]\|_{\text{op}}$ which in turn allows us to set $\sigma^2 := \frac{180b^8}{\nu^4 L}$ for the variance. Now we are ready to apply Theorem 5 which implies

$$\Pr [\lambda_{\min}(\mathcal{P}_T(\mathcal{R} - \mathbb{E}[\mathcal{R}])\mathcal{P}_T) \leq -\delta] \leq d^2 \exp \left(-\frac{\nu^4 \delta^2 L}{\bar{C}_1 b^8} \right)$$

for any $0 \leq \delta \leq 1 < 90b^8/\nu^2 = \sigma^2/\underline{R}$ and \tilde{C}_1 is an absolute constant. This gives a suitable bound on the probability of the undesired event

$$\{\lambda_{\min}(\mathcal{P}_T(\mathcal{R} - \mathbb{E}[\mathcal{R}])\mathcal{P}_T) \leq -\delta\}.$$

If this is not the case, (33) implies

$$(dL)^{-1}\|\mathcal{A}(Z)\|_{\ell_2}^2 > (1 - \delta)\|Z\|_2^2$$

for all matrices $Z \in T$ simultaneously. This proves (32) and setting $\delta = 3/4$ yields Proposition 7 (with $C_1 = \frac{16}{9}\tilde{C}_1$). \square

For our proof we will also require a uniform bound on $\|\mathcal{A}(Z)\|_{\ell_2}$.

Lemma 9 (Robust injectivity, upper bound). *Let \mathcal{A} be as above. Then the statement*

$$(34) \quad \frac{1}{dL}\|\mathcal{A}(Z)\|_{\ell_2}^2 \leq b^4 d\|Z\|_2^2$$

holds with probability 1 for all matrices $Z \in H^d$ simultaneously.

Proof. Estimate

$$\begin{aligned} \frac{1}{dL}\|\mathcal{A}(Z)\|_{\ell_2}^2 &= \frac{1}{dL} \sum_{k,l} (\text{tr}(f_k f_k^* D_l Z D_l))^2 \leq \max_{1 \leq k \leq d} \|f_k f_k^*\|_2^2 \frac{1}{dL} \sum_l \|D_l Z D_l\|_2^2 \\ &\leq d\|D_l\|_\infty^4 \|Z\|_2^2 \leq db^4 \|Z\|_2^2, \end{aligned}$$

where the first inequality holds because the $f_k f_k^*$'s are mutually orthogonal. The second inequality follows from the fact that the Frobenius norm (and more generally: any unitarily invariant norm) is symmetric [33, Proposition IV.2.4] – i.e., $\|ABC\|_2 \leq \|A\|_\infty \|B\|_2 \|C\|_\infty$ for any $A, B, C \in H^d$ – and the last one is due to the a-priori bound $\|D_l\|_\infty \leq b$. \square

5. PROOF OF THE MAIN THEOREM / CONVEX GEOMETRY

In this section, we will prove that the convex program (8) indeed recovers the signal x with high probability. A common approach to prove recovery is to show the existence of an *approximate dual certificate*, which in our problem setup can be formalized by the following definition.

Definition 10 (Approximate dual certificate). *Assume that the sampling process corresponds to (6). Then we call $Y \in H^d$ an approximate dual certificate if $Y \in \text{range } \mathcal{A}^*$ and*

$$(35) \quad \|Y_T - X\|_2 \leq \frac{\nu}{4b^2\sqrt{d}} \quad \text{as well as} \quad \|Y_T^\perp\|_\infty \leq \frac{1}{2}.$$

The following proposition, showing that the existence of such a dual certificate indeed guarantees recovery, is just a slight variation of Proposition 12 in [15]. For completeness, we have nevertheless included a proof in the appendix.

Proposition 11. *Suppose that the measurement gives us access to $\|x\|_{\ell_2}^2$ and $y_{k,l} = |\langle f_k, D_l x \rangle|^2$ for $1 \leq k \leq n$ and $1 \leq l \leq L$. Then the convex optimization (8) recovers the unknown x (up to a global phase), provided that (23) holds and an approximate dual certificate Y exists.*

Proposition 11 proves the Main Theorem of this paper, provided that an approximate dual certificate exists. A first approach to construct an approximate dual certificate is to set

$$(36) \quad Y = \mathcal{R}(X) - \text{tr}(X)\mathbb{1}.$$

Note that any such Y is indeed in the range of our measurement process and, in expectation, yields an exact dual certificate, $\mathbb{E}[Y] = X$. One can then show using an operator Bernstein or Hoeffding inequality that Y is close to its expectation, but the number of measurements required is too large to make the result meaningful. This obstacle can be overcome using the golfing scheme, a refined construction procedure originally introduced in [11].

A main difference between our approach and the approach in [1] is that the authors of that paper use Hoeffding's inequality in the golfing scheme, while we employ Bernstein's inequality. The resulting bounds are sharper, but require to estimate an additional variance parameter.

An issue that remains is that such bounds heavily depend on the worst-case operator norm of the individual summands. In this framework these are proportional to $|\langle f_k, D_l x \rangle|^2$, which a priori can reach $b^2 d$ (recall that $\|f_k\|_2^2 = d$). To deal with this issue, we follow the approach from [15, 1] to condition on the event that their maximal value is not too large.

Lemma 12. *For $Z \in T$ arbitrary and a parameter $\gamma \geq 1$ we introduce the event*

$$(37) \quad U_{k,l} := \left\{ |\text{tr}(F_{k,l}Z)| \leq 2^{3/2} b^2 \gamma \log d \|Z\|_2 \right\},$$

If D_l is chosen according to (3) it holds that

$$\max_{1 \leq k \leq d} \Pr [U_{k,l}^c] \leq 4d^{-\gamma}.$$

In the following, we refer to γ as the *truncation rate* (cf. [15]). Here, we fix

$$(38) \quad \gamma = 8 + \log_2 \frac{b^2}{\nu},$$

for reasons that shall become clear in the proofs of Propositions 15 and 16. Here b and ν are as in (4) and (5).

Proof of Lemma 12. Fix $Z \in T$ arbitrary and apply an eigenvalue decomposition

$$Z = \lambda_1 y y^* + \lambda_2 z z^*$$

with normalized eigenvectors $u, v \in \mathbb{C}^d$. Then one has for $1 \leq k \leq d$:

$$\begin{aligned} \Pr [U_{k,l}^c] &\leq \Pr [|\text{tr}(F_{k,l}Z)| \geq 2b^2 \gamma \log d \|Z\|_1] \\ &\leq \Pr [|\lambda_1| |\langle f_k, D_l y \rangle|^2 + |\lambda_2| |\langle f_k, D_l z \rangle|^2 \geq (|\lambda_1| + |\lambda_2|) 2b^2 \gamma \log d] \\ &\leq \Pr [|\langle f_k, D_l y \rangle| \geq \sqrt{2b^2 \gamma \log d}] + \Pr [|\langle f_k, D_l z \rangle| \geq \sqrt{2b^2 \gamma \log d}], \end{aligned}$$

where the last inequality uses a union bound. The desired statement thus follows from

$$\Pr [|\langle f_k, D_l u \rangle| \geq b \sqrt{2\gamma \log d} \|u\|_{\ell_2}] \leq 2d^{-\gamma} \quad \forall u \in \mathbb{C}^d, \forall 1 \leq k \leq d,$$

which we now aim to show. Fix $1 \leq k \leq d$ and $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ arbitrary and insert the definitions of f_k and D_l to obtain

$$|\langle f_k, D_l u \rangle| = \left| \sum_{i=1}^d \epsilon_i (\omega^{ki} u_i) \right| = \left| \sum_{i=1}^d \epsilon_i \tilde{u}_i \right|.$$

Here we have defined $\tilde{u} = (\omega^k u_1, \dots, \omega^{k(d-1)} u_{d-1}, u_d)$. Note that $\|\tilde{u}\|_{\ell_2} = \|u\|_{\ell_2} = 1$ holds and applying Theorem 3 therefore yields

$$\begin{aligned} \Pr \left[\left| \sum_{i=1}^d \epsilon_i \tilde{u}_i \right| \geq b\sqrt{2\gamma \log d} \right] &= \Pr \left[\left| \sum_{i=1}^d \epsilon_i \tilde{u}_i \right| \geq b\sqrt{2\gamma \log d} \|\tilde{u}\|_2 \right] \\ &\leq 2 \exp(-\gamma \log d) = 2d^{-\gamma}. \end{aligned}$$

□

This result will be an important tool to bound the probability of extreme operator norms.

Definition 13. For $Z \in T$ arbitrary and the corresponding $U_{k,l}$ introduced in (37) we define the truncated measurement operator

$$(39) \quad \mathcal{R}_Z := \sum_{l=1}^L \mathcal{M}_l^Z \quad \text{with} \quad \mathcal{M}_l^Z := \frac{1}{\nu^2 d L} \sum_{k=1}^d 1_{U_{k,l}} \Pi_{F_{k,l}},$$

where $1_{U_{k,l}}$ denotes the indicator function associated with the event $U_{k,l}$.

We now show that in expectation, this truncated operator is close to the original one.

Lemma 14. Fix $Z \in T$ arbitrary and let \mathcal{R}_Z and \mathcal{M}_l^Z be as in (39). Then

$$\begin{aligned} \|\mathbb{E}[\mathcal{R} - \mathcal{R}_Z]\|_{\text{op}} &\leq \frac{4b^4}{\nu^2} d^{2-\gamma} \quad \text{and} \\ \|\mathbb{E}[(\mathcal{M}_l(W))^2 - (\mathcal{M}_l^Z(W))^2]\|_{\infty} &\leq \frac{8b^8}{\nu^4 L^2} d^{4-\gamma} \|W\|_{\infty}^2 \end{aligned}$$

for any $W \in H^d$.

Proof. Note that $\mathbb{E}[\mathcal{R}] = L\mathbb{E}[\mathcal{M}_l]$ as well as $\mathbb{E}[\mathcal{R}_Z] = L\mathbb{E}[\mathcal{M}_l^Z]$. For the first statement, we can therefore fix $1 \leq l \leq L$ arbitrary and consider $L\|\mathbb{E}[\mathcal{M}_l - \mathcal{M}_l^Z]\|_{\infty}$. Inserting the definitions and applying Lemma 12 yields

$$\begin{aligned} &L\|\mathbb{E}[\mathcal{M}_l - \mathcal{M}_l^Z]\|_{\text{op}} \\ &= \left\| \frac{1}{\nu^2 d} \sum_{k=1}^d \mathbb{E}[(1 - 1_{U_{k,l}}) \Pi_{F_{k,l}}] \right\|_{\text{op}} \leq \frac{1}{\nu^2 d} \sum_{k=1}^d \mathbb{E}[1_{U_{k,l}^c} \|\Pi_{F_{k,l}}\|_{\text{op}}] \\ &\leq \frac{b^4 d^2}{\nu^2 d} \sum_{k=1}^d \Pr[U_{k,l}^c] \leq \frac{b^4 d^2}{\nu^2} \max_{1 \leq k \leq d} \Pr[U_{k,l}^c] \leq \frac{4b^4}{\nu^2} d^{2-\gamma}, \end{aligned}$$

where the second inequality is due to $\|\Pi_{F_{k,l}}\|_{\text{op}} \leq b^4 d^2$ (which follows by direct calculation). Similarly

$$\begin{aligned} &\left\| \mathbb{E}[(\mathcal{M}_l(W))^2 - (\mathcal{M}_l^Z(W))^2] \right\|_{\infty} \\ &= \left\| \frac{1}{(\nu^2 d L)^2} \sum_{k,j=1}^d \mathbb{E}[(1 - 1_{U_{k,l}} 1_{U_{j,l}}) \text{tr}(F_{k,l} W) \text{tr}(F_{j,l} W) F_{k,l} F_{j,l}] \right\|_{\infty} \\ &\leq \frac{1}{\nu^4 L^2 d^2} \sum_{k,j=1}^d \mathbb{E}[1_{U_{k,l}^c \cup U_{j,l}^c} |\text{tr}(F_{k,l} W) \text{tr}(F_{j,l} W)| \|F_{k,l}\|_{\infty} \|F_{j,l}\|_{\infty}] \\ &\leq \frac{b^8 d^4}{\nu^4 L^2} \|W\|_{\infty}^2 \max_{1 \leq k,j \leq d} (\Pr[U_{k,l}^c] + \Pr[U_{j,l}^c]) \leq \frac{8b^8}{\nu^4 L^2} d^{4-\gamma} \|W\|_{\infty}^2 \end{aligned}$$

Here we have used $|\operatorname{tr}(F_{k,l}W)| \leq b^2 d \|W\|_\infty$ for any $W \in H^d$ and $\|F_{k,l}\|_\infty \leq b^2 d$ (both estimates are direct consequences of the definition of $F_{k,l}$). \square

We will now establish two technical ingredients for the golfing scheme.

Proposition 15. *Assume $d \geq 3$, fix $Z \in T$ arbitrary and let \mathcal{R}_Z be as in (39). Then*

$$(40) \quad \Pr \left[\|\mathcal{P}_T^\perp(\mathcal{R}_Z(Z) - \operatorname{tr}(Z)\mathbb{1})\|_\infty \geq t\|Z\|_2 \right] \leq d \exp \left(-\frac{t\nu^4 L}{C_2 b^8 \gamma \log d} \right)$$

for any $t \geq 1/4$ and γ defined in (38). Here C_2 denotes an absolute constant.

Proof. Assume w.l.o.g. that $\|Z\|_2 = 1$. By Lemma 6,

$$\mathcal{P}_T^\perp \mathbb{E}[\mathcal{R}(Z)] = \mathcal{P}_T^\perp(Z + \operatorname{tr}(Z)\mathbb{1}) = 0 + \operatorname{tr}(Z)\mathcal{P}_T^\perp \mathbb{1},$$

because $Z \in T$ by assumption. We can thus rewrite the desired expression as

$$\begin{aligned} & \|\mathcal{P}_T^\perp(\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}(Z)])\|_\infty \\ & \leq \|\mathcal{P}_T^\perp(\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)])\|_\infty + \|\mathcal{P}_T^\perp \mathbb{E}[\mathcal{R}_Z(Z) - \mathcal{R}(Z)]\|_2 \\ & \leq \|\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)]\|_\infty + \|\mathbb{E}[\mathcal{R}_Z - \mathcal{R}]\|_{\text{op}} \|Z\|_2 \\ (41) \quad & \leq \|\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)]\|_\infty + t/4. \end{aligned}$$

In the third line, we have used that $\|\mathcal{P}_T^\perp W\| \leq \|W\|$ for any $W \in H^d$ and any unitarily invariant norm $\|\cdot\|$ (pinching, cf. [33] (Problem II.5.4)). The last inequality follows from

$$(42) \quad \|\mathbb{E}[\mathcal{R}_Z - \mathcal{R}]\|_{\text{op}} \leq \frac{4b^4}{\nu^2} d^{2-\gamma} \leq \frac{b^4}{\nu^2} 2^{4-\gamma} \leq \frac{1}{16} \leq \frac{t}{4},$$

which in turn follows from Lemma 14 and the assumptions on d , t and γ . By (41), it remains to bound the probability of the complement of the event

$$E := \{\|\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)]\|_\infty \leq 3t/4\}$$

To this end, we use the Operator Bernstein inequality (Theorem 4). We decompose

$$\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)] = \sum_{l=1}^d (M_l - \mathbb{E}[M_l]) \quad \text{with} \quad M_l := \mathcal{M}_l^Z(Z),$$

where \mathcal{M}_l^Z was defined in (39). To find an a priori bound for the individual summands, we write, using that $F_{k,l} \geq 0$ for all $1 \leq k \leq d$,

$$\begin{aligned} \|M_l\|_\infty & \leq \left\| \frac{1}{\nu^2 d L} \sum_{k=1}^d 1_{U_{k,l}} |\operatorname{tr}(F_{k,l}Z)| F_{k,l} \right\|_\infty \leq \frac{2^{3/2} b^2 \gamma \log d}{\nu^2 L} \|Z\|_2 \|D_l^2\|_\infty \\ (43) \quad & \leq \frac{608 b^8 \gamma \log d}{3 \nu^4 L} =: \overline{R}. \end{aligned}$$

Here we have used that $\frac{1}{d} \sum_{k=1}^d f_k f_k^* = \mathbb{1}$, $\|D_l^2\|_\infty \leq b^2$, and $\nu \leq b^2$. The last estimate is far from tight, but will slightly simplify the resulting operator Bernstein bound. For the variance we start with the standard estimate

$$\mathbb{E}[(M_l - \mathbb{E}[M_l])^2] = \mathbb{E}[M_l^2] - \mathbb{E}[M_l]^2 \leq \mathbb{E}[M_l^2]$$

and bound this expression via

$$\begin{aligned}
\|\mathbb{E}[M_l^2]\|_\infty &= \left\| \mathbb{E}[(\mathcal{M}_l^Z(Z))^2] \right\|_\infty \\
&\leq \left\| \mathbb{E}[(\mathcal{M}_l^Z(Z))^2 - (\mathcal{M}_l(Z))^2] \right\|_\infty + \left\| \mathbb{E}[(\mathcal{M}_l(Z))^2] \right\|_\infty \\
&\leq \frac{8b^8}{\nu^4 L^2} d^{4-\gamma} \|Z\|_\infty^2 + \frac{30b^8}{\nu^4 L^2} \|Z\|_2^2,
\end{aligned}$$

where we have used Lemmas 14 and 8. Using $\|Z\|_\infty \leq \|Z\|_2 = 1$ and noting that $\nu \leq b^2$ entails $\gamma = 8 + 2 \log_2(b^2/\nu) \geq 8$ we conclude

$$\left\| \sum_{l=1}^L \mathbb{E}[M_l^2] \right\|_\infty \leq \sum_{l=1}^L \|\mathbb{E}[M_l^2]\|_\infty \leq \frac{8b^8}{\nu^4 L} d^{-4} + \frac{30b^8}{\nu^4 L} \leq \frac{38b^8}{\nu^4 L} =: \sigma^2.$$

Our choice for \bar{R} now guarantees $\sigma^2/\bar{R} = 3/(16\gamma \log d) \leq 3t/4$ for any $t \geq 1/4$ (here we have used $\gamma \geq 1$ and our assumption $d \geq 3$ which entails $\log d \geq 1$). Consequently

$$\Pr[E^c] = \Pr \left[\left\| \sum_{l=1}^L (M_l - \mathbb{E}[M_l]) \right\|_\infty > 3t/4 \right] \leq d \exp \left(-\frac{t\nu^4 L}{C_2 b^8 \gamma \log d} \right)$$

with C_2 an absolute constant. This completes the proof. \square

Proposition 16. Assume $d \geq 2$ and fix $Z \in T$ arbitrary. Let \mathcal{R}_Z be as in (39), then

$$\Pr[\|\mathcal{P}_T(\mathcal{R}_Z(Z) - Z - \text{tr}(Z)\mathbb{1})\|_2 \geq c\|Z\|_2] \leq 2 \exp \left(-\frac{c\nu^4 L}{C_3 b^8 \gamma \log d} \right)$$

holds for any $c \geq 1/(2 \log d)$ and γ defined in (38). Here, C_3 is again an absolute constant.

Proof. This proof is very similar to the previous one. However, there is one crucial difference: The projection \mathcal{P}_T assures that we only have to deal with rank-2 matrices. Consequently, we can in this case apply the Operator Bernstein inequality to the reduced space T . Again we start by assuming $\|Z\|_2 = 1$ and using *near-isotropy* of \mathcal{R} in order to rewrite the desired expression as

$$\begin{aligned}
&\|\mathcal{P}_T(\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}(Z)])\|_2 \\
&\leq \|\mathcal{P}_T(\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)])\|_2 + \|\mathcal{P}_T \mathbb{E}[\mathcal{R}(Z) - \mathcal{R}_Z(Z)]\|_2 \\
&\leq \sqrt{2} \|\mathcal{P}_T(\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)])\|_\infty + \|\mathbb{E}[\mathcal{R}(Z) - \mathcal{R}_Z(Z)]\|_2 \\
&\leq \sqrt{2} \|\mathcal{P}_T(\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)])\|_\infty + \|\mathbb{E}[\mathcal{R}_Z - \mathcal{R}]\|_{\text{op}} \|Z\|_2 \\
&\leq \sqrt{2} \|\mathcal{P}_T(\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)])\|_\infty + c/4.
\end{aligned}$$

Here we have used $\|\mathcal{P}_T W\|_2 \leq \|W\|_2$ for any matrix W (this follows from the entry-wise definition of the Frobenius norm) and a calculation similar to (42):

$$\|\mathbb{E}[\mathcal{R}_Z - \mathcal{R}]\|_{\text{op}} \leq \frac{4b^4}{\nu^2 d} d^{3-\gamma} \leq \frac{b^4}{\nu^2 \log d} 2^{5-\gamma} \leq \frac{1}{8 \log d} \leq \frac{c}{4},$$

where we have used $d \geq 2$ and $\gamma \geq 8$. Paralleling our idea from the previous proof, we define the event

$$E' := \left\{ \|\mathcal{P}_T(\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)])\|_\infty \leq 3c/(4\sqrt{2}) \right\}$$

which guarantees that the desired inequality is valid. In order to bound the probability of $(E')^c$, we once more aim to apply the Operator Bernstein inequality. Decompose

$$\mathcal{P}_T(\mathcal{R}_Z(Z) - \mathbb{E}[\mathcal{R}_Z(Z)]) = \sum_{l=1}^L \left(\tilde{M}_l - \mathbb{E}[\tilde{M}_l] \right).$$

Note that the \tilde{M}_l 's are related to M_l in the previous proof via

$$\tilde{M}_l = \mathcal{P}_T M_l = \mathcal{P}_T \mathcal{M}_l^Z(Z).$$

We can exploit this similarity together with (22) and (43) to get an a-priori bound:

$$\|\tilde{M}_l\|_\infty = \|\mathcal{P}_T M_l\|_\infty \leq \frac{976\sqrt{2}b^8\gamma \log d}{3\nu^4 L} =: \overline{R}.$$

In the last inequality we have once more used $b^4/\nu^2 \geq 1$. The motivation for such a rather loose estimate is again a slightly simpler operator Bernstein bound. For the variance, we start by noticing

$$\mathbb{E}[(\mathcal{P}_T \mathcal{M}_l^Z(Z))^2] \leq \|\mathbb{E}[(\mathcal{P}_T \mathcal{M}_l(Z))^2 - (\mathcal{P}_T \mathcal{M}_l^Z(Z))^2]\|_\infty + \|\mathbb{E}[(\mathcal{P}_T \mathcal{M}_l(Z))^2]\|_\infty.$$

Paralleling the respective calculation in the proof of Lemma 14 combined with (22) it can be shown that the first term is bounded by $\frac{32b^4}{\nu^2 L^2} d^{-4} \|Z\|_\infty^2$ due to our choice of γ . We can use Lemma 8 to bound the remaining term and obtain

$$\left\| \sum_{l=1}^L \mathbb{E}[\tilde{M}_l^2] \right\|_\infty \leq \sum_{l=1}^L \|\mathbb{E}[\tilde{M}_l^2]\|_\infty \leq \frac{32b^4}{\nu^2 L} d^{-4} \|Z\|_\infty^2 + \frac{90b^8}{\nu^4 L} \|Z\|_2^2 \leq \frac{122b^8}{\nu^4 L} =: \sigma^2.$$

Consequently $\sigma^2/\overline{R} \leq 3/(8\sqrt{2} \log d) \leq 3c/(4\sqrt{2})$ for any $c \geq 1/(2 \log d)$. Therefore we need to consider the second special case of Theorem 4 applied to the two-dimensional matrix space T . Applying it yields the desired bound on $\Pr[E'^c]$. \square

We are now ready to construct a suitable approximate dual certificate in the sense of Definition 10. The key idea here is an iterative procedure – dubbed the *golfing scheme* – that was first established in [11] (see also [34, 20, 1, 15]).

Proposition 17. *Assume $d \geq 3$ and let $\omega \geq 1$ be arbitrary. If the total number of L of diffraction patterns fulfills*

$$(44) \quad L \geq C\omega \log^2(d),$$

then with probability larger than $1 - 5/6e^{-\omega}$, an approximate dual certificate Y as in Definition 10 can be constructed using the golfing scheme. Here, C is an absolute constant that only depends on the probability distribution used to generate the random masks D_l .

To be concrete, the constant C depends on the truncation rate γ and the a-priori bound b and ν of the random variable ϵ used to generate the diffraction patterns D_l :

$$C = \tilde{C} \frac{b^8}{\nu^4} \log \frac{b^2}{\nu},$$

where \tilde{C} is an absolute constant.

Proof. This construction is inspired by [20] and [34]. As in [11] our construction of Y follows a recursive procedure of w iterations. The i -th iteration depends on three parameters $L_i \in \mathbb{N}$ and $t_i \geq 1/4$ as well as $c_i \geq 1/(2 \log d)$ which will be specified later on. To initialize, set

$$Y_0 = 0$$

and define, for $Y_i, i \geq 1$, defined recursively below,

$$Q_i := X - \mathcal{P}_T Y_i \in T.$$

The i -th step proceeds according to the following protocol:

We sample L_i masks D_1, \dots, D_{L_i} independently according to (3). Let $\tilde{\mathcal{R}}_{Q_{i-1}}$ be the measurement operator of length L_i introduced in Definition 13 (so that the summands are conditioned on $U_{k,l}$ for $Q_{i-1} \in T$). Then we check whether for our choice of b_i, c_i the inequalities

$$(45) \quad \|\mathcal{P}_T^\perp \left(\tilde{\mathcal{R}}_{Q_{i-1}}(Q_{i-1}) - \text{tr}(Q_{i-1})\mathbb{1} \right)\|_\infty \leq t_i \|Q_{i-1}\|_2 \quad \text{and}$$

$$(46) \quad \|\mathcal{P}_T(\tilde{\mathcal{R}}_{Q_{i-1}}(Q_{i-1}) - Q_{i-1} - \text{tr}(Q_{i-1})\mathbb{1})\|_2 \leq c_i \|Q_{i-1}\|_2$$

are satisfied for that particular $\tilde{\mathcal{R}}_{Q_{i-1}}$. If so, set $\mathcal{R}_{Q_{i-1}}^{(i)} = \tilde{\mathcal{R}}_{Q_{i-1}}$ as well as

$$Y_i = \mathcal{R}_{Q_{i-1}}^{(i)}(Q_{i-1}) - \text{tr}(Q_{i-1})\mathbb{1} + Y_{i-1}$$

and proceed to step $(i+1)$. If either (45) or (46) fails to hold, repeat the i -th step with of L_i different masks drawn independently according to (3). We denote the probability of having to repeat the i -th step by $p_{\text{err}}(i)$ and the eventual number of repetitions by $r_i \geq 1$.

Then one obtains (cf. [20, Lemma 14] for details):

$$\begin{aligned} Y &:= Y_w = \mathcal{R}_{Q_{w-1}}^{(w)}(Q_{w-1}) - \text{tr}(Q_{w-1})\mathbb{1} + Y_{w-1} \\ &= \sum_{i=1}^w \left(\mathcal{R}_{Q_{i-1}}^{(i)}(Q_{i-1}) - \text{tr}(Q_{i-1})\mathbb{1} \right) \quad \text{and} \\ Q_i &= X - \mathcal{P}_T Y_i = \mathcal{P}_T \left(Q_{i-1} + \text{tr}(Q_{i-1})\mathbb{1} - \mathcal{R}_{Q_{i-1}}^{(i)}(Q_{i-1}) \right) \\ &= \dots = \prod_{j=1}^i \mathcal{P}_T(\mathcal{I} + \Pi_{\mathbb{1}} - \mathcal{R}_{Q_{j-1}}^{(j)})Q_0. \end{aligned}$$

The validity of properties (45) and (46) in each step now guarantees

$$\begin{aligned} \|Y_T - X\|_2 &= \|Q_w\|_2 \leq \prod_{j=1}^w c_j \|Q_0\|_2 = \prod_{i=1}^w c_i \|X\|_2 = \prod_{i=1}^w c_i, \\ \|Y_T^\perp\|_\infty &\leq \sum_{i=1}^w \left\| \mathcal{P}_T^\perp \left(\mathcal{R}_{Q_{i-1}}^{(i)}(Q_{i-1}) - \text{tr}(Q_{i-1})\mathbb{1} \right) \right\|_\infty \\ &\leq \sum_{i=1}^w t_i \|Q_{i-1}\|_2 \leq t_1 + \sum_{i=2}^w t_i \prod_{j=1}^{i-1} c_j. \end{aligned}$$

Inspired by [20], we now set the parameters

$$w = \frac{1}{2} \lceil \log_2 d \rceil + \lceil 2 \log_2 b \rceil + 2 - \lfloor \log_2 \nu \rfloor, \quad c_1 = \frac{1}{2 \log d}, \quad t_1 = \frac{1}{4}$$

and for $i \geq 2$

$$c_i = \frac{1}{2}, \quad t_i = \frac{\log d}{4}.$$

Note that the overall dimension d and all these parameters obey the conditions required for Propositions 15 and 16, respectively. These constants now assure

$$\begin{aligned} \|Y_T - X\|_2 &\leq \prod_{i=1}^w c_i = \frac{1}{\log d} 2^{-w} \leq \frac{\nu}{4b^2\sqrt{d}}, \\ \|Y_T^\perp\|_\infty &\leq t_1 + \sum_{i=2}^w t_i \prod_{j=1}^{i-1} c_j = \frac{1}{4} + \sum_{i=2}^w \frac{\log d}{4} \frac{1}{\log d} 2^{1-i} \leq \frac{1}{4} + \frac{1}{4} \sum_{i=1}^\infty 2^{-i} = \frac{1}{2}, \end{aligned}$$

which are precisely the requirements (35) on Y .

All that remains to be done now is to estimate the probability that the total number of measurements

$$L = \sum_{i=1}^w L_i r_i$$

exceeds the bound given in (44). More precisely – as in [20] – we will bound the probability

$$p_{\text{err}} := \Pr \left[(r_1 \geq 1), \text{ or } \sum_{i=2}^w r_i L_i \geq w' \right]$$

for some “oversampling” w' to be chosen later. Disregarding the first step (which is special in our iterative procedure) for the moment, it is useful to think of a random walk which advances from position i to $(i+1)$ if a newly sampled batch fulfills inequalities (45, 46); and remains at position i if this is not the case, i.e., with probability $p_{\text{err}}(i)$. Note that according to the union bound we have

$$p_{\text{err}}(i) \leq \Pr[(45) \text{ fails to hold in step } i] + \Pr[(46) \text{ fails to hold in step } i].$$

In that sense, p_{err} is the probability that either the first step fails, or that the random walker fails to reach position w before exceeding the allowed number of trials.

To obtain concrete numbers, choose

$$L_1 = C_4 \frac{b^8}{\nu^4} \omega \gamma \log^2 d \quad \text{and for } i \geq 2 \quad L_i = C_5 \frac{b^8}{\nu^4} \gamma \log d,$$

where C_4, C_5 are numerical constants sufficiently large (in particular $C_4, C_5 \geq \max\{C_2, C_3\}$) to guarantee via Propositions 15 and 16:

$$\begin{aligned} p_{\text{err}}(1) &\leq 2 \exp \left(-\frac{(2 \log d)^{-1} \nu^4 L_1}{C_3 b^8 \gamma \log d} \right) + d \exp \left(-\frac{4^{-1} \nu^4 L}{C_2 b^8 \gamma \log d} \right) \\ &\leq \frac{1}{3} e^{-\omega} \quad \text{and} \\ p_{\text{err}}(i) &\leq 2 \exp \left(-\frac{2^{-1} \nu^4 L_i}{C_3 b^8 \gamma \log d} \right) + d \exp \left(-\frac{(\log d/4) \nu^4 L_i}{C_2 b^8 \gamma \log d} \right) \leq \frac{1}{20}. \end{aligned}$$

In the total number (44) of masks, the numerical constant C is assumed to be large enough such that

$$L \geq L_1 + (2w + 3\omega + 6 \log 2) L_i$$

holds. This assures, that one can in fact (over-) sample

$$(47) \quad w' := 2w + 3\omega + 6 \log 2$$

batches after the first two. Applying the union bound to the total failure probability p_{err} reveals

$$p_{\text{err}} \leq p_{\text{err}}(1) + \Pr \left[\sum_{i=2}^w r_i L_i \geq w' \right].$$

The last expression is in one-to-one correspondence with the probability that fewer than $w-1$ successes occur in $w' \geq w$ trials with individual failure probability smaller than $1/20$. The latter probability can be estimated using a standard concentration bound for binomial random variables, e.g.

$$\Pr [|\text{Bin}(n, p) - np| \geq \tau] \leq 2 \exp \left(-\frac{\tau^2}{3np} \right)$$

from [35, Section Concentration]. In this particular situation, $n = w'$, $p = 19/20$ and $\tau = (w' - w + 1)$ is adequate. The choice of w' in (47) then assures

$$\begin{aligned} \Pr \left[\sum_{i=3}^w r_i L_i \geq w' \right] &\leq \Pr [|\text{Bin}(w', 19/20) - 19w'/20| \geq w' - w + 1] \\ &\leq 2 \exp \left(-\frac{20(w' - w + 1)^2}{3 \times 19w'} \right) \leq \frac{1}{2} e^{-\omega}. \end{aligned}$$

This together with our bound (47) on $p_{\text{err}}(1)$ assures

$$p_{\text{err}} \leq \frac{1}{3} e^{-\omega} + \frac{1}{2} e^{-\omega} = \frac{5}{6} e^{-\omega}.$$

However, this is just the performance guarantee which we require in Proposition 17. \square

We now have all the ingredients for the proof of our main result, Theorem 1.

Proof of the Main Theorem. With probability at least $1 - 5/6 e^{-\omega}$, the construction of Proposition 17 yields an approximate dual certificate provided that

$$L \geq C \frac{b^8}{\nu^4} \gamma \omega \log^2 d,$$

where C is a sufficiently large constant. In addition, by Proposition 7, one has (23) with probability at least $1 - 1/6 e^{-\omega}$, potentially with an increased value of C . Thus the result follows from Proposition 11 and a union bound over the two probabilities of failure. \square

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6. APPENDIX

Proof of Lemma 6. We prove formula (18) in a way that is slightly different from the proof provided in [1]. We show that the set of all possible $D_l f_k$ ’s is in fact proportional to a 2-design and deduce *near-isotropy* of \mathcal{R} from this. We refer to [15] for further clarification of the concepts used here. Concretely, for $1 \leq l \leq L$ we aim to show

$$(48) \quad \frac{1}{\nu^2 d} \sum_{k=1}^d \mathbb{E} [F_{k,l}^{\otimes 2}] = 2P_{\text{Sym}^2},$$

where P_{Sym^2} denotes the projector onto the totally symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$. Near isotropy of \mathcal{R} directly follows from (48) by applying [29, Lemma 1] (with $\alpha = \beta = 1$):

$$\mathbb{E} [\mathcal{R}] Z = \frac{1}{\nu^2 d L} \sum_{k=1}^d \sum_{l=1}^L \mathbb{E} [F_{k,l} \text{tr}(F_{k,l} Z)] = \frac{1}{\nu^2 d} \sum_{k=1}^d \mathbb{E} [F_{k,1} \text{tr}(F_{k,1} Z)] = (\mathcal{I} + \Pi_1) Z.$$

So let us proceed to deriving equation (48). We do this by exploring the action of the equation’s left hand side on a tensor product $e_i \otimes e_j$ ($1 \leq i, j \leq d$) of two standard basis vectors in \mathbb{C}^d . Here it is important to distinguish two special cases, namely $i = j$ and $i \neq j$. For the former we get by inserting standard basis representations

$$\begin{aligned} \frac{1}{\nu^2 d} \sum_{k=1}^d \mathbb{E} [F_k^{\otimes 2}] (e_i \otimes e_i) &= \frac{1}{\nu^2 d} \sum_{k=1}^d \mathbb{E} [\epsilon_i^2 \langle f_k, e_i \rangle^2 D^{\otimes 2}(f_k \otimes f_k)] \\ &= \frac{1}{\nu^2} \sum_{a,b=1}^d \mathbb{E} [\epsilon_i^2 \epsilon_a \epsilon_b] \left(\frac{1}{d} \sum_{k=1}^d \omega^{k(u+v-2i)} \right) (e_a \otimes e_b) \\ &= \frac{1}{\nu^2} \sum_{a,b=1}^d \delta_{(a \oplus b), (2i)} \mathbb{E} [\epsilon_i^2 \epsilon_a \epsilon_b] (e_a \otimes e_b). \end{aligned}$$

Now, $\mathbb{E}[\epsilon_i] = 0$ demands that the indices a and b have to “pair up” (i.e., $a = b$ must hold) in order to yield a non-vanishing contribution. Therefore one in fact gets

$$\frac{1}{\nu^2 d} \sum_{k=1}^d \mathbb{E} [F_k^{\otimes 2}] (e_i \otimes e_i) = \frac{1}{\nu^2} \sum_{a=1}^d \delta_{(2a), (2i)} \mathbb{E} [\epsilon_a^2 \epsilon_a^2] (e_a \otimes e_b) = \frac{1}{\nu^2} \mathbb{E} [\epsilon_i^4] (e_i \otimes e_i) = 2(e_i \otimes e_i),$$

where we have used the moment condition (5) in the last step. This however is equivalent to the action of $2P_{\text{Sym}^2}$ on symmetric basis states.

Let us now focus on the second case, namely $i \neq j$. A similar calculation then yields

$$\frac{1}{\nu^2 d} \sum_{k=1}^d \mathbb{E} [F_k^{\otimes 2}] (e_i \otimes e_j) = \frac{1}{\nu^2} \sum_{a,b=1}^d \mathbb{E} [\epsilon_i \epsilon_j \epsilon_a \epsilon_b] \delta_{(u+v), (i+j)} (e_a \otimes e_b).$$

Again, $\mathbb{E}[\epsilon] = 0$ demands that the ϵ ’s have to “pair up”. Since $i \neq j$ by assumption, there are only two such possibilities, namely $(i = a, j = b)$ and $(i = b, j = a)$. Both pairings obey the additional delta-constraint and we therefore get

$$\frac{1}{\nu^2 d} \sum_{k=1}^d \mathbb{E} [F_k^{\otimes 2}] (e_i \otimes e_j) = \frac{1}{\nu^2} \mathbb{E} [\epsilon_i^2 \epsilon_j^2] (e_i \otimes e_j + e_j \otimes e_i) = (e_i \otimes e_j) + (e_j \otimes e_i),$$

where we have once more used (5) in the final step. This, however is again just the action of $2P_{\text{Sym}^2}$ on vectors $e_i \otimes e_j$ with $i \neq j$. Since the extended standard basis $\{(e_i \otimes e_j)\}_{1 \leq i, j \leq d}$ forms a complete basis of $\mathbb{C}^d \otimes \mathbb{C}^d$, we can deduce equation (48) from this. \square

Proof of Proposition 11. Let X' be an arbitrary feasible point of (8) and we decompose it as $X' = X + \Delta$, where Δ is a feasible displacement. Feasibility then implies $\mathcal{A}(X') = \mathcal{A}(X)$ and consequently $\mathcal{A}(\Delta) = 0$ must hold. The pinching inequality [33] (Problem II.5.4) now implies

$$\|X'\|_1 = \|X + \Delta\|_1 \geq \|X\|_1 + \text{tr}(\Delta_T) + \|\Delta_T^\perp\|_1$$

and X is guaranteed to be the minimum of (8) if

$$(49) \quad \text{tr}(\Delta_T) + \|\Delta_T^\perp\|_1 > 0$$

is true for any feasible displacement Δ . Therefore it suffices to show that (49) is guaranteed to hold under the assumptions of the proposition. In order to do so, we combine feasibility of Δ with Proposition 7 and Lemma 9 to obtain

$$(50) \quad \|\Delta_T\|_2 < \frac{2}{\sqrt{\nu^2 d L}} \|\mathcal{A}(\Delta_T)\|_{\ell_2} = \frac{2}{\nu \sqrt{d L}} \|\mathcal{A}(\Delta_T^\perp)\|_{\ell_2} \leq \frac{2b^2 \sqrt{d}}{\nu} \|\Delta_T^\perp\|_2.$$

Feasibility of Δ also implies $(Y, \Delta) = 0$, because $Y \in \text{range}(\mathcal{A}^*)$ by definition. Combining this insight with (50) and the defining property (35) of Y now yields

$$\begin{aligned} 0 &= (Y, \Delta) = (Y_T - X, \Delta_T) + (X, \Delta_T) + (Y_T^\perp, \Delta_T^\perp) \\ &\leq \|Y_T - X\|_2 \|\Delta_T\|_2 + \text{tr}(\Delta_T) + \|Y_T^\perp\|_\infty \|\Delta_T^\perp\|_1 \\ &< \text{tr}(\Delta_T) + \|Y_T - X\|_2 2b^2 \sqrt{d}/\nu \|\Delta_T^\perp\|_2 + \|Y_T^\perp\|_\infty \|\Delta_T^\perp\|_1 \\ &\leq \text{tr}(\Delta_T) + 1/2 \|\Delta_T^\perp\|_2 + 1/2 \|\Delta_T^\perp\|_1 \\ &\leq \text{tr}(\Delta_T) + \|\Delta_T^\perp\|_1, \end{aligned}$$

which is just the optimality criterion (49). \square